Fujita Approximation: Everything today / $\mathbb{C}$.

If D is an integral divisor on a normal variety X of dim n

then $\mathrm{vol}(D) := \limsup_m \dfrac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!} \in \mathbb{R}_{\geq 0}$.

Ex: $\mathrm{vol}(\mathcal{O}_{\mathbb{P}^n}(1)) = 1$

$\mathrm{vol}(D) > 0$ iff D is big.

Facts: • volume is a numeric invariant: if $D.C = D'.C$ for all curves C, then $\mathrm{vol}(D) = \mathrm{vol}(D')$.

• volume is defined on $N^1(X) = NS/\mathrm{num}$.

• $\mathrm{vol}(pD) = p^n \mathrm{vol}(D) \longrightarrow$ know how to extend to $\mathbb{Q}$-divisors

• $\mathrm{vol}: N^1(X) \longrightarrow \mathbb{R}_{\geq 0}$ is continuous (but not a "nice" function) when D is nef & big, $\mathrm{vol}(D) = (D)^n$.

(This follows from asymptotic RR for $\chi(\mathcal{O}_X(mD))$,

plus fact that if D nef then $h^i(X, \mathcal{O}_X(mD)) \sim m^{n-1}$

for $i > 0$)

in particular: choose mD s.t. it defines a rational map.

Taking n general elements $D_1, \dots, D_n \in |mD|$, we just

have $\mathrm{vol}(D) = D^n = \# D_1 \cap \dots \cap D_n$.

$\longrightarrow$ $\mathrm{vol}(D)$ is an integer (if D is integral)

In general, things are more complicated:

volume of an integral divisor can be arbitrarily small

Ex: $a \in \mathbb{N}$ let C be a curve of genus ≥ 1, ~~take~~ $x \in C$,

let $\mathcal{E} = \mathcal{O}_C((1-a)x) \oplus \mathcal{O}_C(x)$. If $X = \mathbb{P}(\mathcal{E})$, ~~then~~ not nef

$\mathrm{vol}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = 1/a$.

in fact, $\mathrm{vol}(D)$ for D an integral divisor can be irrational. (ex: E an elliptic curve, $E \times E$

X a $\mathbb{P}^1$-bundle over $E \times E$, $\mathrm{vol}(\mathcal{O}_X(1))$ is irrational)

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in general, studying volume of non-nef divisors is harder.

<u>Theorem</u>: (Fujita) if $\xi \in N^1(X)$ is big, and we choose $\varepsilon > 0$, then there's $\mu : X' \longrightarrow X$ proper & birational, s.t. $\mu^*(\xi) = a + e$, w/ a is ample, e is effective, and $\mathrm{vol}(a) > \mathrm{vol}(\xi) - \varepsilon$. (note: $\mathrm{vol}(\xi) \geq \mathrm{vol}(a)$)

$\mathrm{vol}(\xi) = \mathrm{vol}(\mu^*\xi)$

and: if ξ is a $\mathbb{Q}$-divisor, we can take a & e to be $\mathbb{Q}$ as well.

equiv: if L is a $\mathbb{Q}$-divisor, $\varepsilon>0$, $\exists\ \mu: X' \to X$

s.t. $\mu^*(pL) = A + E$, w/ A, E integral divisors

A ample

E effective,

$\in \mathbb{Z}^+$

$$(A)^n = \mathrm{vol}(A) > p^n(\mathrm{vol}(L) - \varepsilon)$$

it's equivalent to ask that A is big & nef, rather than ample.

claim: if A is big & nef, there's an effective divisor F s.t. $A - F/k$ is ample for all k.

vol is continuous, so for $k \gg 0$, $\mathrm{vol}(A) \simeq \mathrm{vol}(A - E/k)$

$$A + E = \underbrace{(A - F/k)}_{\text{ample}} + \underbrace{(F/k + E)}_{\text{effective}}$$

if $\mu : X' \longrightarrow X$ is a Fujita approx., w/ $\mu^*(pL) = A + E$, then for any further $\pi : X'' \longrightarrow X'$, $\pi^*\mu^*(pL) = \pi^*A + \pi^*E$

$\mathrm{vol}(\pi^*A) = \mathrm{vol}(A)$.

So, π^*A is big & nef (so can perturb to make ample).

<u>So</u>: can take X' to be smooth, or dominating another birational morphism to X, or a simultaneous resolution of several divisor classes.

<u>proof of claim:</u> D nef & big.

D big $\Longrightarrow$ $mD =_{lin} A + E$, w/ A ample, E eff. for $m \gg 0$ fixed.

so, $kD = A + E + (k-m)D$. If $k - m > 0$,

then $kD = \underbrace{\left(A + (k-m)D\right)}_{\substack{\text{ample + nef} \\ = \text{ample}}} + \underbrace{E}_{\text{eff.}}$

$D - E/k = \left.\frac{1}{k}\left(A + (k-m)D\right)\right]$ ample.

for $k \gg 0$.

Consequences:

- $\mathrm{vol}(D) := \limsup_{m} \frac{h^0(\mathcal{O}_X(mD))}{m^n/n!} = \lim_{m\to\infty} \frac{h^0(\mathcal{O}_X(mD))}{m^n/n!}$

- $\mathrm{vol}(A+B)^{\frac{1}{n}} \geq \mathrm{vol}(A)^{\frac{1}{n}} + \mathrm{vol}(B)^{\frac{1}{n}}$
  "log concavity of volume"
  idea! easy to prove for ample divisors.
  Fujita allows to extend to arbitrary big divisors.

- $\mathrm{vol}(D)$ is given by "moving intersection number".
  Define $(mD)^{[n]}$ by ~~taking~~ $D_1, \dots, D_n \in |mD|$, and setting $(mD)^{[n]} = \#(D_1 \cap \dots \cap D_n - \mathrm{Bs}|mD|)$.
  then: $\mathrm{vol}(D) = \lim_{m\to\infty} \frac{(mD)^{[n]}}{m^n}$

EX: Let $X = Bl_p \mathbb{P}^2$. Say H is pullback of hyperplane class in $\mathbb{P}^2$, E exc. divisor.

Consider $H + aE$. not nef: $(H+aE).E = -a$.

$(H+aE)^2 = 1 - a^2$. $\quad h^0(m(H+aE)) = h^0(mH + maE)$

$= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$

$\mathrm{vol}(H+aE) = \mathrm{vol}(H) = 1$.

take $D_1, D_2 \in |m(H+aE)| = |mH + maE|$.

D_1, D_2 look like transform of general degree-m curves in $\mathbb{P}^2$ plus maE.

$\# \, D_1 \cap D_2 - \underbrace{Bs(|mD|)}_{E} = m^2$. $\qquad \dfrac{(mD)^{[2]}}{m^2} = 1$ for all m.

pf of Fujita approx:

2 facts about multiplier ideals. M b.s

- global generation: let M be a divisor, and let B very ample. Then $\mathcal{O}_X(K_X + (n+1)B + \ell M) \otimes \mathcal{J}(\|\ell M\|)$ is globally generated. (follows from Nadel vanishing)

- subadditivity: $\mathcal{J}(\|\ell M\|) \subseteq \mathcal{J}(\|M\|)^{\ell}$.

- recall: $H^0(\mathcal{O}_X(\ell M)) = H^0(\mathcal{O}_X(\ell M) \otimes \mathcal{J}(\|\ell M\|))$

By continuity, can assume we're approximating a $\mathbb{Q}$-divisor class, say L is an integral divisor,
Fix ε,
we want $\mu : X' \longrightarrow X$ proper birational,

$\mu^*(pL) = A + E$, w/ A ~~ample~~ big & nef.
E effective,

and $\mathrm{vol}(A) > p^n(\mathrm{vol}(L) - \varepsilon)$.

idea of proof: take M to be a slight perturbation of pL, let $\mathcal{J} = \mathcal{J}(\|M\|)$. Let

$\mu : X' = Bl_{\mathcal{J}} X' \longrightarrow X$; we'll show this satisfies above properties.

we can resolve singularities & X is smooth.

choose B a very ample divisor, ample enough that $G = K_X + (n+1)B$ is very ample as well.

Choose $p \gg 0$ s.t.
- $pL - G$ is big.
- $\operatorname{vol}(pL - G) > p^n(\operatorname{vol}(L) - \varepsilon)$.

Set $M = pL - G$. Let $\mathcal{J} = \mathcal{J}(\|M\|)$.

$$\mathcal{O}_X(pL) \otimes \mathcal{J} = \underbrace{\mathcal{O}_X(K_X + (n+1)B + M) \otimes \mathcal{J}(\|M\|)}_{\text{this is globally generated.}}$$

so, $\mathcal{O}_X(pL) \otimes \mathcal{J}$ is globally generated

Let $\mu : X' \longrightarrow X$ be the (normalized) blowup along $\mathcal{J}$. say $\mathcal{J} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$

$$\underbrace{\mathcal{O}_{X'}(\mu^*(pL) - E)}_{\text{globally generated, hence nef.}} = \mu^*(\mathcal{O}_X(pL) \otimes \mathcal{J})$$

$\mu^*(pL) - E$ is nef. Set $A = \frac{1}{p}(\mu^*(pL) - E))$.

$$\mu^*(L) = \underbrace{A}_{\text{nef}} + \underbrace{E/p}_{\text{eff}}$$ is our "prospective" Fujita approx.

just need $\mathrm{vol}(A) > \mathrm{vol}(L) - \varepsilon$.

claim: for any ℓ, $H^0(X, \mathcal{O}_X(\ell M)) \subset H^0(X, \mathcal{O}_X(\ell p L) \otimes \mathcal{J}^\ell)$

(recall: $M = pL - G$)

$$H^0(\mathcal{O}_X(\ell M)) = H^0(\mathcal{O}_X(\ell M) \otimes \mathcal{J}(\|\ell M\|))$$

$$\overset{\text{subadditivity}}{\subseteq} H^0(\mathcal{O}_X(\ell M) \otimes \mathcal{J}(\|M\|)^\ell)$$

$\ell M = \ell p L - \ell G$, so $\mathcal{O}_X(\ell M) \subset \mathcal{O}_X(\ell p L)$

Thus, $H^0(\mathcal{O}_X(\ell M) \otimes \mathcal{J}^\ell) \subset H^0(\mathcal{O}_X(\ell p L) \otimes \mathcal{J}^\ell)$.

takeaway: $H^0(\mathcal{O}_X(\ell M)) \subset H^0(\mathcal{O}_X(\ell p L) \otimes \mathcal{J}^\ell)$

want to show: $\boxed{\mathrm{vol}(pA) > \mathrm{vol}(M).}$

$A = \frac{1}{p}(\mu^*(pL) - E)$

<u>chose</u> $M = pL - G$ s.t. $\mathrm{vol}(M) > p^n(\mathrm{vol}(L) - \varepsilon)$

$$H^0(\mathcal{O}_X(lM)) \subseteq H^0(\mathcal{O}_X(lpL) \otimes \mathcal{J}^l)$$

$$= H^0(X', \mathcal{O}_{X'}(\mu^*(lpL) - lE))$$

$$(\mu_* \mathcal{O}_{X'}(\mu^*(lpL) - lE) = \mathcal{O}_X(lpL) \otimes \mathcal{J}^l)$$

$$\mu^*(lpL) - lE = lpA$$

$$H^0(X, \mathcal{O}_X(lM)) \leq H^0(X', \mathcal{O}_{X'}(lpA)).$$

Thus, $\mathrm{vol}(M) \leq \mathrm{vol}(pA).$

Thus: $\quad \mu^*(\rho L) = \left(\mu^*_{pA}(\rho L) - E\right) + E$ effective

(where $\quad \mathcal{J}\cdot\theta_{x'} = \theta_{x'}(-E))$) bg & ref

is our Fujita approx.